Classical stability of $U(1)_A$ domain walls in dense matter QCD

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It was recently shown that there exists metastable $U(1)_A$ domain wall configurations in high-density QCD ($\mu \gg 1$ GeV). We will assess the stability of such nontrivial field configurations at intermediate densities ($\mu < 1$ GeV). The existence of such configurations at intermediate densities could have interesting consequences for the physics of neutron stars where such densities are realized.

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I. INTRODUCTION

In general, there do not exist domain walls or other topological defects within the standard model. This is due to the trivial topology of the vacuum manifold. In contrast, these objects are quite common in condensed-matter physics and cosmology. However, it has been realized only recently [1–4] that topological defects such as domain walls and strings may exist within the standard model at a large chemical potential.

It is well known that topological defects result from symmetries being broken. In the past few years there has been renewed interest in high-density QCD. Similar to the BCS pairing in conventional superconductivity, the ground state of QCD at high density is unstable due to the formation of a diquark condensate [5–7] (see [8] for a review). In this new ground state, various symmetries which are present at $\mu = 0$ are broken by the presence of this nonzero diquark condensate. This leads to the formation of the various topological defects discussed in [1–4].

In [1], it was shown that at high densities ($\mu \gg 1$ GeV) there exist domain wall solutions which interpolate between the same vacuum state, in which the $U(1)_A$ phase of the diquark condensate varies between 0 and 2π. This type of domain-wall which interpolates between the same vacuum state has been studied before in the context of axion models [9]. It is interesting to note that similar domain-wall configurations are present for the zero-density case in the large-N_c limit and could be present for the physically relevant case of $N_c = 3$ [10], in which case they can be studied at the BNL Relativistic Heavy Ion Collider (RHIC) [10,11]. Given this, one might ask what happens between these two regions of $\mu = 0$ and $\mu \gg 1$ GeV. The main goal of the present paper is the analysis of the classical stability of the $U(1)_A$ domain-walls for $\mu \lesssim 1$ GeV. We demonstrate that the $U(1)_A$ domain-walls are classically stable down to densities $\mu \simeq 800$ MeV. The main idea is to interpolate between $\mu \gg 1$ GeV (where perturbation theory is justified) and $\mu \simeq \mu_c$ (where instanton calculations lead to a reasonable description). Ideally, we would like to be able to make definitive statements on the stability of such configurations all the way down from large μ to $\mu_c \sim 500$ MeV, where μ_c is the critical chemical potential above which the color superconducting phase occurs. Unfortunately, we are unable to make a definitive statement for $\mu_c \lesssim \mu \lesssim 800$ MeV due to a lack of theoretical control in this region. One can only speculate on the behavior in this region.

This work is organized as follows. In Sec. II, we will construct the effective potential. In Sec. III, we will describe the nontrivial domain-wall solutions. In Sec. IV, we will examine the classical stability of such configurations under small perturbations. In Sec. V, we will end with concluding remarks and future considerations.

II. THE EFFECTIVE POTENTIAL

It is well known that the ground state of $N_f = 2,3$, $N_c = 3$ QCD exhibits the Cooper pairing phenomenon as in conventional superconductivity [known as the color 2SC (2 flavor color superconducting) and CFL (color-flavor locked) phases of QCD [6,7,12,13]]. In what follows, we will consider the $N_c = N_f = 3$ CFL phase. In the CFL phase, the condensates take the form

$$\langle q^{ia}_{L\alpha} q^{jb}_{L\beta} \rangle^* \sim \epsilon_{\alpha\beta\gamma} \epsilon^{ij} \epsilon^{abc} X^{\gamma}_{c},$$
$$\langle q^{ia}_{R\alpha} q^{jb}_{R\beta} \rangle^* \sim \epsilon_{\alpha\beta\gamma} \epsilon^{ij} \epsilon^{abc} Y^{\gamma}_{c}, \quad (1)$$

where L and R represent left- and right-handed quarks; α, β, and γ are the flavor indices; i and j are spinor indices; a,b, and c are color indices; and X^{γ}_{c} and Y^{γ}_{c} are the condensates which are complex color-flavor matrices. As the magnitudes of these condensates depend on the color index c, one can easily see that these objects are not gauge-invariant by themselves. In order to construct a gauge-invariant field, the following matrix which describes the octet of mesons and axial singlet was considered in [14,15]:

$$\Sigma^{\beta}_{\gamma} = X Y^{\dagger} = \sum_{c} X^{\beta}_{c} Y^{c*}_{\gamma}. \quad (2)$$

If a $U(1)_A$ rotation ($q \rightarrow \exp^{i\gamma_5 \alpha/2} q$) of this gauge-invariant field Σ is performed, we see that the fields (1) transform as

$$X \rightarrow e^{-i\alpha} X,$$
$$Y \rightarrow e^{+i\alpha} Y, \quad (3)$$

and therefore

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$$\Sigma \rightarrow e^{-2i\alpha}\Sigma. \tag{4}$$

Goldstone's theorem states that there must be a single Goldstone mode η' associated with the breaking of this symmetry. Given this, we can parametrize the field as follows:

$$\Sigma = \Sigma_o e^{\rho}\ e^{-i\phi}, \tag{5}$$

where the phase $\phi = \eta'/f$ is defined as a dimensionless field which describes the η' boson, f is the corresponding decay constant, Σ_o is the vacuum expectation value of the composite field (2), and ρ is another dimensionless field which describes the fluctuations of the magnitude of the condensate[1] (analogous to the σ field related to the fluctuations of the $\langle \bar{q}q \rangle$ chiral condensate). We choose to parametrize the field as e^{ρ} for convenience later on.

In order to construct an effective potential describing the dynamics of the phase of the condensate ϕ as well as the magnitude $|\Sigma|$, there are two types of terms which must be included. The first term which explicitly breaks the $U(1)_A$ symmetry was calculated in [1,16] by substituting the form of condensates given above into the instanton-induced four-fermion Lagrangian [17,18]:

$$V_{1\text{-inst}}(\rho,\phi) = -a\mu^2\Delta_o^2 e^{\rho}\cos\phi, \tag{6}$$

where Δ_o is the value of the gap in the quark spectrum. The perturbative form of the expectation value of the condensate has been used in arriving at this result [1,19]:

$$\langle|\Sigma|\rangle \equiv \Sigma_o = \frac{9}{8\pi^2}\frac{\mu^4\Delta_o^2}{g^2}. \tag{7}$$

For $N_f = 3$, the dimensionless coefficient a was found to be

$$a = 7\times 10^3 \left(\frac{m_s}{\mu}\right)\left(\ln\frac{\mu}{\Lambda_{\rm QCD}}\right)^7\left(\frac{\Lambda_{\rm QCD}}{\mu}\right)^9, \tag{8}$$

where m_s is the mass of the strange quark. The mass of the corresponding η' boson can be easily calculated by expanding the potential, $m_{\eta'} = 2\pi\sqrt{a}\ \Delta_o$. According to [1], this potential is only under theoretical control when the mass of the η' boson is much less than the typical scale for higher excitations 2Δ, which corresponds to $a \ll 1/\pi^2$. For physical values of the strange quark, this corresponds to a chemical potential of about $\mu \approx 700$ MeV (for $a \sim 1/\pi^2$).

To be able to draw any conclusions about the stability of the domain-wall solution for intermediate densities ($\mu_c \lesssim \mu \lesssim 1$ GeV), one must include degrees of freedom which are related to the fluctuations of the absolute value of $|\Sigma|$. We do not know the effective potential in the region of interest; however, for qualitative discussions we shall use a potential derived for asymptotically large μ. The second type of term which must be included in an effective potential description is one which uniquely fixes the magnitude of the condensate. An effective potential for the magnitude of the condensate $|\Sigma|$ was derived in [20] in the perturbative region where the analytical form of the gap is known. This effective potential fixes uniquely the value of the vacuum condensate. The potential is of the Coleman-Weinberg type [21] and is given as follows:

$$\begin{aligned} V_{\rm pert}(|\Sigma|) &= -\left(\frac{2\nu\pi}{\mu}\right)^2|\Sigma|\left(1-\ln\frac{|\Sigma|}{\Sigma_o}\right), \\ &= -\frac{\mu^2\Delta_o^2}{\pi^2}e^{\rho}(1-\rho), \\ \nu &= \sqrt{\frac{8\alpha_s}{9\pi}}, \end{aligned} \tag{9}$$

where α_s is the standard strong-coupling constant and the perturbative result of the condensate (7) has been used. In all calculations that follow we will assume $\Delta_o = 100$ MeV as the numerical value for the gap. It should be noted that the potential (9) is justified only in the region

$$\nu \log(|\Sigma|/\Sigma_o) \ll 1. \tag{10}$$

Since we are considering $|\Sigma| \approx \Sigma_o$, the use of this effective potential is justified.

We are interested in the region $\mu_c \lesssim \mu \lesssim 1$ GeV, where Eq. (9) is not literally correct. Even though the results stated above are not necessarily under theoretical control for $\mu \gtrsim \mu_c$, one can speculate on how the coefficients behave at intermediate densities when $\mu_c \lesssim \mu \lesssim 1$ GeV. Due to the fact that all the same symmetries are present as the chemical potential is lowered until reaching μ_c, we expect the qualitative form of the effective potential (6) and (9) to remain the same. As the chemical potential is lowered, eventually perturbative calculations which are valid at asymptotically large μ are no longer the correct description and instanton calculations become relevant. One would expect that these calculations must match up at some point. However, the coefficients in front of the potential could possibly be very different in the density region of interest. We will refer to the coefficient in front of the one-instanton potential as β_1 and the coefficient in front of the perturbative potential as β_2. The value of β_1 is essentially fixed by the form of the condensate and by the constituent quark mass. Below the critical chemical potential μ_c at which the chiral phase transition occurs, this coefficient is independent of μ [13]. Therefore, we would expect that β_1 would be some smooth function of μ which reaches its maximum value at $\mu = \mu_c$. At $\mu_c \lesssim \mu \lesssim 1$ GeV, the coefficient β_2 is modified by the formation of instanton–anti-instanton ($I\bar{I}$) molecules [13]. We should also note that the coefficient β_1 can also be estimated from the instanton liquid model [13] using the average size of instantons as well as requiring a constituent quark mass of about 350–400 MeV.

Combining both terms we have an effective potential which is given by the following:

[1]The field ρ should not be confused with the familiar ρ meson in QCD.

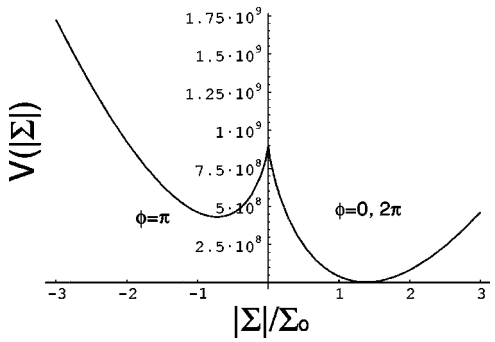

FIG. 1. The cross section of the effective potential is shown above for $\mu=800$ MeV. On the left half $\phi=\pi$ is shown and on the right half $\phi=0,2\pi$ is shown.

$$V(\rho,\phi)=-\beta_2 e^{\rho}(1-\rho)-\beta_1 e^{\rho}\cos\phi. \tag{11}$$

The values of the coefficients β_1 and β_2 are known for asymptotically large μ:

$$\beta_1=a\mu^2\Delta_o^2, \tag{12}$$

$$\beta_2=\frac{\mu^2\Delta_o^2}{\pi^2}. \tag{13}$$

In Fig. 1, we show the cross section of the potential at $\phi=0,2\pi$ and $\phi=\pi$ for $\mu=800$ MeV. Notice that the existence of an absolute minima at $\phi=0,2\pi$ and a saddle point at $\phi=\pi$ allows for nontrivial configurations which wind around the barrier at $|\Sigma|=0$. In the limit $\mu\to\infty$, the parameter $a\to 0$ and the potential has degenerate minima at $|\Sigma|=\Sigma_o$. The kinetic term is given by

$$\frac{|\partial_o\Sigma|^2-u^2|\partial_i\Sigma|^2}{|\Sigma|^2}=(\partial_o\rho)^2-u^2(\partial_i\rho)^2 +(\partial_o\phi)^2-u^2(\partial_i\phi)^2, \tag{14}$$

where u is the velocity which is different from 1. The perturbative values for the decay constant f and velocity u were calculated in [22]. In order to fix the correct dimensionality, we must multiply the kinetic term by the appropriate powers of the decay constant. The full effective Lagrangian up to two derivatives in the fields is then given by

$$\mathcal{L}=f^2[(\partial_o\rho)^2-u^2(\partial_i\rho)^2] +f^2[(\partial_o\phi)^2-u^2(\partial_i\phi)^2]-V(\rho,\phi). \tag{15}$$

In the above we have assumed that $f_{\eta'}\simeq f_\rho=f$. The exact numerical value for f_ρ is not known. However, in the large-μ limit $f_{\eta'}\sim f_\rho\sim\mu$ in order to have an appropriate scale for $m_\rho\sim\Delta_o$ (once again, the ρ field should not be confused with the well-known ρ meson in QCD at $\mu=0$).

III. DOMAIN-WALL SOLUTIONS

As was done in [1], if we replace the field $|\Sigma|=\Sigma_o e^{\rho}$ by its vacuum expectation value (which is justified for $\mu\gg 1$ GeV), the resulting potential is of the sine-Gordon type. The Lagrangian is given by the following:

$$\mathcal{L}=f^2[(\partial_0\phi)^2-u^2(\partial_i\phi)^2]-V_{1\text{-inst}}(\phi), \tag{16}$$

where the constant term has been dropped. The static domain-wall solution to the corresponding equation of motion is well known. Considering a domain-wall in the z direction, the solution is given by

$$\phi_o(z)=4\ \arctan(\exp(-mz/u)), \tag{17}$$

where m is the mass of the η'. This solution interpolates between the same vacuum state; at $z=\pm\infty$ we have $\phi=0,2\pi$, respectively. It is well known that this solution is absolutely stable under small perturbations $\phi=\phi_o+\delta\phi$. In other words, the Schrödinger-type equation obtained by varying the field and linearizing the equation of motion,

$$-\partial_z^2\ \ \delta\phi_n+\frac{m^2}{u^2}\left[1-2\ \mathrm{sech}^2\left(\frac{m}{u}z\right)\right]\delta\phi_n=\omega_n^2\delta\phi_n, \tag{18}$$

has the lowest eigenvalue $\omega_o=0$ corresponding to $\delta\phi_o=d\phi_o(z)/dz\sim\mathrm{sech}(mz/u)$. This is just the zero mode which is a result of translational invariance $z\to z+z_o$. Since the lowest eigenvalue is non-negative, the domain-wall solution is stable under small perturbations. It turns out that this is the only bound state which is a solution of Eq. (18).

In the case in which the replacement $|\Sigma|\to\Sigma_o$ is not done, the solution must be modified. If we want to study a stable solution for μ which is not asymptotically large, we must include fluctuations in ϕ as well as in the ρ direction (i.e., the absolute value of Σ). The two equations of motion for static solutions are given by

$$2f^2u^2\nabla_i^2\rho=\beta_2\ \ \rho\ \ e^{\rho}-\beta_1 e^{\rho}\cos\phi, \tag{19}$$

$$2f^2u^2\nabla_i^2\phi=\beta_1 e^{\rho}\sin\phi. \tag{20}$$

Although we do not know the exact solution for this set of coupled nonlinear differential equations, if $\beta_1/\beta_2<1$ we can approximate the solutions. In this case, the approximate solutions can be parametrized by

$$e^{\rho_o}\approx 1+\alpha\cos\phi_o, \tag{21}$$

$$\phi_o\approx 4\arctan(\exp(-mz/u)), \tag{22}$$

where $\alpha\approx\beta_1/\beta_2$. Our stability analysis will be based upon these approximate solutions of the equations of motion. Since the above solutions do not correspond to the exact solutions to the equations of motion (minimum energy path which winds around that barrier at $|\Sigma|=0$), there will be nonzero linear terms when the energy of the system is perturbed about the domain-wall solutions ϕ_o and ρ_o. These will be estimated in the following section where the stability analysis is performed.

IV. STABILITY ANALYSIS OF DOMAIN WALLS

Although these domain walls may exist as classically stable objects at large densities, it is not immediately obvious if this is the case at intermediate densities. In order to examine the classical stability of the domain-wall configurations in the region $\mu \lesssim 1$ GeV, we must look at how the system reacts to small perturbations of the fields. The energy density is given by the following expression:

$$E(\phi,\rho)=\int_{-\infty}^{+\infty} dz[f^2u^2(\nabla\rho)^2 + f^2u^2(\nabla\phi)^2+V(\phi,\rho)]. \tag{23}$$

In order to express all integrals as dimensionless quantities, we will perform the following change of variables:

$$z'=\frac{z}{\lambda}, \quad \lambda=\frac{u}{m_{\eta'}}. \tag{24}$$

The energy density is now given by

$$E(\phi,\rho)=\frac{f^2u^2}{\lambda}\left[\int dz'\left((\nabla\rho)^2+(\nabla\phi)^2 + \frac{\lambda^2}{f^2u^2}V(\phi,\rho)\right)\right], \tag{25}$$

where the derivative is now taken with respect to the dimensionless coordinate z'. We can see that this has the correct dimensions of MeV^3 times the dimensionless integral in square brackets. Following the standard method for analyzing the stability of a classical solution which was briefly described in the previous section, we will expand the fields about their vacuum expectation values:

$$\rho\rightarrow\rho_o+\delta\rho, \qquad \phi\rightarrow\phi_o+\delta\phi. \tag{26}$$

Next, we substitute Eq. (26) into Eq. (23) and perform an expansion, neglecting any terms greater than quadratic order in $\delta\rho$ and $\delta\phi$:

$$\begin{aligned} E &\simeq E^{(0)}+E^{(1)}+E^{(2)} \\ &\simeq E(\phi_o,\rho_o)+\gamma\int dz'\,\delta\rho\left(-2\nabla^2\rho_o+\frac{\lambda^2}{f^2u^2}\frac{\delta V}{\delta\rho}\bigg|_{\phi_o,\rho_o}\right)+\gamma\int dz'\,\delta\phi\left(-2\nabla^2\phi_o+\frac{\lambda^2}{f^2u^2}\frac{\delta V}{\delta\phi}\bigg|_{\phi_o,\rho_o}\right) \\ &\quad+\gamma\int dz'\,\delta\rho\left(-\nabla^2+\frac{\lambda^2}{2f^2u^2}\frac{\delta^2V}{\delta\rho^2}\bigg|_{\phi_o,\rho_o}\right)\delta\rho+\gamma\int dz'\,\delta\phi\left(\frac{\lambda^2}{f^2u^2}\frac{\delta^2V}{\delta\phi\,\delta\rho}\bigg|_{\phi_o,\rho_o}\right)\delta\rho \\ &\quad+\gamma\int dz'\,\delta\phi\left(-\nabla^2+\frac{\lambda^2}{2f^2u^2}\frac{\delta^2V}{\delta\phi^2}\bigg|_{\phi_o,\rho_o}\right)\delta\phi, \end{aligned} \tag{27}$$

where $\gamma=f^2u^2/\lambda$. The first term $E^{(0)}$ in the above expansion is the energy density or wall tension of the domain-wall. In the case in which the domain-wall solutions given by Eq. (21) were the exact solutions to the classical equations of motion, the linear terms $E^{(1)}$ (proportional to $\delta\rho$ and $\delta\phi$) would be zero everywhere. Due to the fact that our solutions are not exact, these must be considered.

First, let us estimate the term which is linear in $\delta\phi$:

$$\delta\phi^{(1)}=\frac{f^2u^2}{\lambda}\int dz'\,\delta\phi\left(-2\nabla^2\phi_o+\frac{\lambda^2}{f^2u^2}\beta_1 e^{\rho_o}\sin\phi_o\right). \tag{28}$$

We know that $e^{\rho_o}\approx 1+\rho_o$ and using the fact that ϕ_o is a solution to the equation of motion given by Eq. (20) with $\rho=0$, we have

$$\begin{aligned} \delta\phi^{(1)} &\approx \frac{f^2u^2}{\lambda}\int dz'\ \frac{\lambda^2}{f^2u^2}\beta_1\rho_o\sin\phi_o\delta\phi \\ &\approx \frac{f^2u^2}{\lambda}\int dz'\,a\pi^2\sin 2\phi_o\delta\phi. \end{aligned} \tag{29}$$

This linear term goes like $a\pi^2$ (~ 0.3 for $\mu=800$ MeV) and the integrand is small compared to the wall tension $E^{(0)}$ and can be neglected. The term which is linear in $\delta\rho$ is

$$\delta\rho^{(1)}=\frac{f^2u^2}{\lambda}\int dz'\,\delta\rho\left(-2\nabla^2\rho_o+\frac{\lambda^2}{f^2u^2}\beta_2\rho_o e^{\rho_o} - \frac{\lambda^2}{f^2u^2}\beta_1 e^{\rho_o}\cos\phi_o\right). \tag{30}$$

Using again that $\rho_o\approx(\beta_1/\beta_2)\cos\phi_o$, we see this simplifies to

$$\delta\rho^{(1)} \approx \frac{f^2u^2}{\lambda}\int dz'[-2a \quad \pi^2\delta\rho\nabla^2(\cos\phi_o)]. \tag{31}$$

Since this term is also proportional to $a\pi^2$, $\delta\rho \ll \rho_o$, and the integral of $\nabla^2(\cos\phi_o)$ is small, we can also neglect this term. The magnitude of the linear term shows how far away we are from the exact solution. This information will be used in what follows for the stability analysis.

Now we consider the most important quadratic term. A similar case involving two coupled scalar fields was looked at in [23] and we will follow the standard procedure presented there closely. If the field configuration is classically stable, the second variation of the energy should be a positive differential operator. This means we must solve the following Schrödinger-type eigenvalue problem:

$$H\begin{pmatrix}\delta\rho\\ \delta\phi\end{pmatrix} = \omega^2\begin{pmatrix}\delta\rho\\ \delta\phi\end{pmatrix}, \tag{32}$$

where H is the operator,

$$H = -\partial_z^2,\mathbf{1} + \frac{\lambda^2}{2f^2u^2}U, \tag{33}$$

and $\mathbf{1}$ is the 2×2 identity matrix. The potential U is a 2×2 matrix with elements:

$$U_{11} = \beta_2 \quad (1+\rho_o)e^{\rho_o} - \beta_1 e^{\rho_o}\cos\phi_o, \tag{34}$$

$$U_{12} = U_{21} = \beta_1 e^{\rho_o}\sin\phi_o, \tag{35}$$

$$U_{22} = \beta_1 e^{\rho_o}\cos\phi_o. \tag{36}$$

If the domain-wall solution is a stable one, then the operator H is positive-semidefinite. The eigenvalue equations can be decoupled by diagonalizing the matrix U. We should note that only the potential term has to be diagonalized when looking for negative energy modes, as was done in [23]. The result is

$$U_{\pm} = \frac{1}{2}[a \pm \sqrt{a^2 - 4(ab - b^2 - c^2)}], \tag{37}$$

where a,b, and c are defined as

$$\begin{aligned} a &= \beta_2(1+\rho_o) \quad e^{\rho_o},\\ b &= \beta_1 e^{\rho_o}\cos\phi_o,\\ c &= \beta_1 e^{\rho_o}\sin\phi_o. \end{aligned} \tag{38}$$

The operator H now takes the following form:

$$H = \begin{pmatrix} -\partial_{z'}^2 + \dfrac{\lambda^2}{2f^2u^2}U_+ & 0 \\ 0 & -\partial_{z'}^2 + \dfrac{\lambda^2}{2f^2u^2}U_- \end{pmatrix}. \tag{39}$$

Since we can immediately see that $U_+ \geqslant 0$ for all z' due to the fact that $a > 0$, there does not exist any negative eigenvalues corresponding to the first equation in this transformed basis. Looking at the second eigenvalue equation, we have

$$\left(-\partial_{z'}^2 + \frac{\lambda^2}{2f^2u^2}U_-\right)\psi_n = \omega_n^2\psi_n. \tag{40}$$

It is a well-known theorem of quantum mechanics that there must exist at least one bound-state solution to Eq. (40). Due to the fact that our domain-wall solution should be invariant under translations in space $z' \to z' + z_o'$, there should be a corresponding zero mode in the spectrum of Eq. (40). In the high-density limit, we recover the familiar sine-Gordon equation and we know that there is only one bound state in the spectrum of Eq. (18). If the exact solution to the equations of motion (19) and (20) were known, one would expect to see a corresponding mode with a vanishing eigenvalue in the spectrum of Eq. (40). As the density is lowered ($\beta_1 \leqslant \beta_2$) and the saddle point at $\phi = \pi$ is still present, one would expect the appearance of a mode with a negative eigenvalue corresponding to instability of the domain-wall. Due to the fact that there still must be a zero mode in the spectrum, the zero mode would become the first excited state of Eq. (40) and the lowest mode would have some negative eigenvalue $\omega_0^2 < 0$ corresponding to the instability of the domain-wall. The problem of stability analysis now reduces to determining the eigenvalues corresponding to the bound states of Eq. (40). The appearance of an additional bound state in the spectrum as the chemical potential is lowered will be the first sign that the system is approaching the point of instability.

Since the solution corresponding to Eq. (21) is not the exact solution but does represent a path which winds around the barrier at $|\Sigma| = 0$, it is quite possible that the zero mode could show up in the spectrum with a small nonzero eigenvalue. It would show up as a true zero mode only when the exact solution to the equation of motion is substituted into Eqs. (38).

Although the potential U_- is nontrivial, we will use a variational approach in order to determine the upper bounds on ω_0^2 and ω_1^2. In choosing a trial wave function, we make the observation that the potential U_- is quite similar to the same potential which arises when analyzing the stability of the sine-Gordon soliton, Eq. (18). We will pick our normalized trial wave function accordingly:

$$\psi_0 = \sqrt{\frac{\sigma}{2}}\ \mathrm{sech}(\sigma z'), \tag{41}$$

with σ being the variational parameter. Note that this trial wave function satisfies the required boundary conditions $\psi_o(z' = \pm\infty) \to 0$. For the first excited state, we must pick an odd function of z. We will choose the properly normalized function

$$\psi_1 = \sqrt{\frac{3\sigma}{2}}\tanh(\sigma z')\mathrm{sech}(\sigma z'). \tag{42}$$

Applying the variational principle, we must calculate

$$E_n^{(2)}(\sigma) = \frac{f^2u^2}{\lambda}\langle\psi_n|\left(-\partial_{z'}^2 + \frac{\lambda^2}{2f^2u^2}U_-\right)|\psi_n\rangle, \tag{43}$$

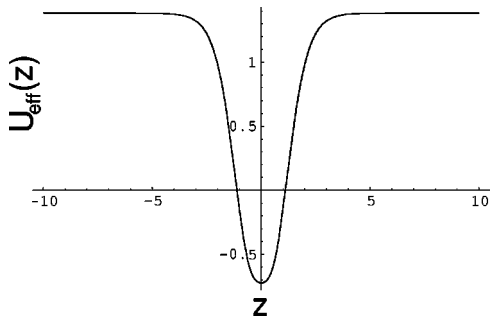

FIG. 2. This figure shows the effective Schrödinger potential $U_{\rm eff}(z')=(\lambda^2/2f^2u^2)U_-(z')$ as a function of the dimensionless coordinate z' where U_- is given by Eq. (37). The presence of a negative eigenvalue in the spectrum of this potential is indicative of the instability of the classical domain-wall solution. The effective potential is shown in this figure for $\mu=800$ MeV.

and minimize this quantity with respect to σ to obtain an upper bound $E_n^{(2)}(\sigma)$ on the energy of the nth state. The integral given by Eq. (43) must be performed numerically.

We now have the framework in place in order to test the stability of our domain-wall configurations. We will assume the perturbative value for the constants f and u as calculated in [22], $f^2=\mu^2/(8\pi^2)$ and $u^2=\frac{1}{3}$. Setting $\mu=800$ MeV and $m_s=100$ MeV, we see that the ratio of the coefficients is $\beta_1/\beta_2\approx 0.3$. In this case, the linear terms (29) and (31) are small and the solutions given by Eq. (21) are valid approximations to the exact solutions. In Fig. 2 we show the effective Schrödinger potential $U_{\rm eff}=\lambda^2U_-/(2f^2u^2)$ for $\mu=800$ MeV. For the above choice of parameters, the wall tension given by Eq. (25) is

$$E^{(0)}(\phi_o,\rho_o)\approx 17.48\frac{f^2u^2}{\lambda}. \tag{44}$$

For the trial wave functions given in Eqs. (41) and (42), the following results for the two bound states of Eq. (40) were obtained:

$$E_0^{(2)}(\sigma_{\rm min})\leqslant -0.016\frac{f^2u^2}{\lambda},$$

$$E_1^{(2)}(\sigma_{\rm min})\leqslant +1.163\frac{f^2u^2}{\lambda}. \tag{45}$$

From this, we can see that $E_1^{(2)}\gg E_0^{(2)}$ and both of these quantities are much less than the wall tension given by Eq. (44). Even though the ground-state energy seems to be negative, due to the fact that $E_1^{(2)}\gg E_0^{(2)}$ we can associate this mode with the zero mode. The small nonzero eigenvalue is actually an artifact of our approximations. The appearance of a negative mode is merely a consequence of the approximate solutions (21) as discussed above. This identification can be verified by increasing the chemical potential. When the above calculations are repeated as the chemical potential is increased, we see that the energy $E_0^{(2)}\to 0$. This result is expected due to the fact that the $|\Sigma|$ field can be integrated out as μ increases and the sine-Gordon-type theory is recovered. As μ is increased, we also see that the eigenvalue of the first excited state ω_1 increases towards the maximum value of $U_{\rm eff}$. Eventually, as μ is increased further, there is no longer a first excited bound state in the spectrum. The calculation of $E_0^{(2)}$ and $E_1^{(2)}$ was done with the variational functions chosen to be different from Eqs. (41) and (42). The results obtained were the same order of magnitude as stated above Eq. (45). This supports our interpretation of the $E_0^{(2)}$ state as the would-be translational mode if exact solutions are known. The magnitudes of the linear terms (29) and (31) are approximately the same order of magnitude of $E_0^{(2)}$, which supports our interpretation of $E_0^{(2)}$ as the zero mode.

V. CONCLUSION

The main goal of this paper was an analysis of the classical stability of $U(1)_A$ domain walls [1]. Naively one would expect that decreasing μ from $\mu\gg 1$ GeV (when the calculations are under control [1]), we inevitably face the situation in which the domain walls become unstable objects due to the fast growth of the coefficient β_1 (11). This naive expectation may not necessarily be correct due to the even faster growth of the coefficient β_2 (11), which receives contributions from the formation of $I\bar{I}$ molecules as well as perturbative contributions.

What we have actually demonstrated is that the domain-wall solution remains classically stable down to $\mu\simeq 800$ MeV. In order to assess the stability of the domain-walls for $\mu_c\lesssim\mu\lesssim 800$ MeV, one must explicitly include the $I\bar{I}$ contribution in the effective potential (which is expected to be the dominant contribution at $\mu\sim\mu_c$ [13]). This would hopefully lead to $\beta_1/\beta_2<1$, ensuring the classical stability of the $U(1)_A$ domain-walls. Unfortunately, due to the lack of information in this region we cannot generalize our results to below 800 MeV to μ_c. We can argue that the $U(1)_A$ domain-walls remain classically stable down to $\mu\gtrsim\mu_c$ due to the faster growth of the coefficient β_2 compared to β_1 as μ is decreased. The ratio β_1/β_2 may be very sensitive to changes in the instanton size distribution or the various form factors. This is a difficult problem and unfortunately I do not know how to estimate such contributions for a small chemical potential.

We should remark here that the stability of the $U(1)_A$ domain-wall implies a classical stability of the $U(1)_A$ strings [3], which become the edge of the domain-walls. It remains to be seen whether these (or other) topological defects will have any impact on the physics of neutron stars and other compact stellar objects with high core density μ.

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